USAAAO 2018 Second Round Solutions

March-April 2018

1 Short Questions

1. In order to detect an Earth-twin, we need significant advances in the precision of spectrographs to detect the periodic Doppler shift of nearby stars. Estimate the radial velocity semi-amplitude, in m/s, that a planet with the mass, radius, and semi-major axis of Earth would cause in the motion of a star with the mass of the Sun. Assume that the Earth-twin has zero eccentricity. Note that the mass of Earth is $5.97 \cdot 10^{24}\, kg$, and the distance from the Earth to the Sun is $1.5 \cdot 10^{8}\, km$.

Solution: Since $v_p = \frac{2\pi a}{T} = 29.9\, km/s$, and from 2^{nd} Kepler's law $\frac{v_s}{v_p} = \frac{m_p}{m_s}$. Calculating it ends up being

$$v_s = v_p \cdot \frac{m_p}{m_s} = 29.9 km/s \cdot \frac{M_e}{M_s},$$

where M_e is the mass of the Earth and M_s is the mass of the Sun. Plugging in the numbers we get $v_s = 0.090 m/s$. Thus the radial velocity semi-amplitude of $\sim 0.1 m/s$.

2. Planet nine is a hypothesized planet in the outer Solar System that may explain the clustering of orbital elements of distant trans-Neptunian objects. The hypothesized periapse of planet nine is $200 AU$, and the apoapse is expected to be at approximately $1200 AU$. What would the eccentricity of planet nine be? How does this eccentricity compare to that of the 8 major planets in the Solar System?

Solution: $e = \frac{r_a - r_p}{(r_a + r_p)} = .714$, much larger than any planet in the Solar System.

3. What is the main-sequence lifetime of a star with a mass of 0.1 Solar masses, and a star with a mass of 10 Solar masses? Assume that stellar luminosity, $L \propto M3.5$, where M is stellar mass, and that the main-sequence lifetime of the Sun is 10 billion years.

Solution: If $L \propto M3.5$, then age $t \propto M^{-2.5}$. Plugging in for $M = 10$ and 0.1 solar masses and $t_0 = 10 Gyr$, we get:

$t = 31.6$ million years for $M = 10$ solar masses

$t = 3160$ billion years for $M = 0.1$ solar masses.

4. The Very Large Array radio interferometer ($\lambda = 1m$) has maximum baseline of $D = 36.4 km$. How large will an optical telescope have to be to achieve a similar angular resolution in visible light ($\lambda = 5,500 \mathring{A}$)?

Solution: Answer: $20mm$

$$\Theta[rad] = 1.22\frac{\lambda}{Diameter} = 1.22\frac{1}{36.4 \cdot 10^3} = 0.0000335rad$$

$$Diameter = 1.22\frac{\lambda}{\Theta} = 1.22\frac{550 \cdot 10^{-9}}{0.0000335} = 0.020m = 20mm$$

5. An amateur astronomer observes the Moon with $20cm$ telescope, and accomplishes 160x magnification with an eyepiece with focal length $10mm$. What is the f-number of the telescope?

Solution: Answer: 8

$Magnification = \frac{F}{f}$, where F is telescope focal length, f is eyepiece focal length

$$F = Magnification \cdot f = 160 \cdot 10 = 1600mm$$

$$F - number = \frac{F}{D} = \frac{1600}{200} = 8$$

6. The average person has $1.4m^2$ of skin. What is the energy per second radiated by the average person in the form of blackbody radiation? What is the peak wavelength of emitted radiation? Why can't we see it with our eyes?

Solution: Answer: $L = A \cdot \sigma \cdot T^4 = 1.4 \cdot 5.6710^{-8} \cdot (273 + 37)^4 = 733W$

Note that the answer may vary slightly $(700 - 730W)$ on the assumed temperature of the human body. In the above example, we have chosen $37C$.

$\Lambda[\mathring{A}] = 2.9 \cdot \frac{10^7}{T[K]} = 93548\mathring{A} = 9.4microns \implies$ IR (human eye does not see in IR)

7. On March 21^{st} at true noon, length of the shadow of a vertical rod was equal to its height. On which geographic latitude did this happen?

Solution: Answer: $\phi = \pm 45$.

21^{st} of March is the day of spring equinox. Thus the Sun can have a height of $h = 90 - \phi$. Since the length of the shadow is equal to its height it means that height of the Sun was $45°$. Thus geographic latitude of this place was $\phi = \pm 45°$(both north and south count).

8. In stars like the Sun, helium nuclei are formed by fusing hydrogen nuclei together in a process known as the proton-proton chain. One step of the proton-proton chain consists of a deuterium nucleus ($m_d =$ 2.01410 u) fusing together with a hydrogen nucleus (m_H = 1.00783 u) to form a helium-3 nucleus (m_{He} = 3.01603 u), where $u = 1.6605 \times 10^{-27}kg$. How much energy is released during this fusion reaction?

Solution: It is necessary to calculate the mass defect. The missing mass has been converted into energy. From conservation of mass,

$$m_d + m_H = m_{He} + \Delta m.$$

Rearranging,

$$\Delta m = m_d + m_H - m_{He},$$

and plugging in the given masses,

$$\Delta m = (2.01410 + 1.00783 - 3.01603)\ \mathrm{u},$$

$$\Delta m = 0.0059\ \mathrm{u},$$

$$\Delta m = (0.0059\ \mathrm{u})\left(1.6605 \times 10^{-27}\ \frac{\mathrm{kg}}{\mathrm{u}}\right),$$

$$\Delta m = 9.797 \times 10^{-}30\ \mathrm{kg}.$$

Then to find the energy,

$$E = mc^2,$$

$$E = (9.797 \times 10^{-}30\ \mathrm{kg})\left(3.0 \times 10^{8}\ \frac{\mathrm{m}}{\mathrm{s}}\right)^2,$$

$$E = 8.817 \times 10^{-13}\ \mathrm{J}.$$

9. Solar wind consists of protons that fly with the speed of 300 km/s and they fill the space of interplanetary matter around Earth with 10 particles/$cm3$. With what force is this "wind" hitting the Moon? Recall that mass of a proton is $m_p = 1.6 \cdot 10^{24}\, g$. Radius of the Moon is $R_m = 1737\, km$.

Solution: Answer: $F = 1.4$ tons.

From 2^{nd} Newton's Law $F = a \cdot m = \frac{\Delta V}{\Delta t} m = \frac{\Delta (Vm)}{\Delta t}$, i.e. equals change of impulse per unit of time. We assume that the protons reaching to the Moon give its impuls without changing its mass. Let V be the wind speed and ρ the density of particles. Then ρV particles hit per unit time per unit area of the Moon, giving impulse $m_p V \rho V$. Over the whole surface of the Moon we get $F = \pi R_m^2 \rho m_p V^2 = 1.4$ tons.

10. Mars orbits the Sun at an average distance of 2.28×10^{11} m and has a radius of 3.39×10^{6} m. The Sun has a luminosity of 3.828×10^{26} W. How much solar energy falls on the surface of Mars each second? Ignore any effects of Mars' thin atmosphere.

Solution: At the distance of Mars' orbit, the Sun's energy output is spread over a sphere with an area of

$$A = 4\pi r^2 = 4\pi(2.28 \times 10^{11} \text{ m})^2 = 6.53 \times 10^{23} \text{ m}^2.$$

Dividing the luminosity of the Sun by this area gives,

$$\frac{L_\odot}{A} = \frac{3.828 \times 10^{26} \text{ W}}{6.53 \times 10^{23} \text{ m}^2} = 586 \ \frac{\text{W}}{\text{m}^2}.$$

Mars presents a circular area to the Sun of

$$A_{\text{Mars}} = \pi r_{\text{Mars}}^2 = \pi(3.39 \times 10^6 \text{ m})^2 = 3.61 \times 10^{13} \text{ m}^2.$$

Therefore, the total energy that falls on the surface of Mars will be

$$\frac{L_\odot}{A} A_{\text{Mars}} = \left(586 \ \frac{\text{W}}{\text{m}^2}\right)(3.61 \times 10^{13} \text{ m}^2) = 2.12 \times 10^{16} \text{ W}.$$

11. When a gravitationally bound system (such as a galaxy) forms, it transitions from a just bound state ($E_{kin} = |E_{pot}|$) to a virialized state ($E_{kin} = 0.5|E_{pot}|$) and the excess binding energy has to be radiated away. Consider an idealized disk galaxy with an exactly flat rotation curve with a rotation speed of $v_{circ} = 220 km/s$ (you can neglect the kinetic energy in random motions). Its density profile cuts off abruptly at a radius of $R_{max} = 50 kpc$. Assume that it took 500 million years for this galaxy to collapse to its present state. What was its mean luminosity (in units of solar luminosity) due to the release of the binding energy during that period?

Solution: Since the final kinetic energy is only half of the potential energy, the amount radiated away must be also equal to the current kinetic energy.

First we need to estimate the total mass of the galaxy. The easiest way to do this is to look at the centripetal force acting on a particle at the outermost radius:

$$\frac{mv_{circ}^2}{R_{max}} = \frac{GmM}{R_{max}^2} \rightarrow M = \frac{v_{circ}^2}{GR_{max}^2}.$$

The outer radius is 50 kpc and $v_{circ} = 220 \ km/s$, so $M = 1.12 \times 10^{45} \ g = 5.63 \times 10^{11} M_\odot$. The kinetic energy is

$$\frac{1}{2} M v_{circ}^2 = 2.71 \times 10^{59} \ erg.$$

The mean luminosity is just

$$L = \frac{E_{kin}}{t} = 1.715 \times 10^{43} ergs^{-1} = 4.48 \times 10^9 L_\odot.$$

2 Long Questions

1. In 2008, while observing WASP-14, a main sequence star of mass 1.211 $M_\odot$ and radius 1.306 $R_\odot$, an exoplanet called WASP-14b was discovered via the transit method. Photometry as well as radial velocity data are shown in the figures. Transits occur once every 2.243753 days. The radial velocity of the center

of mass of WASP-14 and its planet is -4.99 km/s. Fitting of the radial velocity curve indicates that the argument of periastron of the orbit of WASP-14b is 254.9°.

(a) Determine the length of the semimajor axis of the orbit of WASP-14b.

Solution: Answer: $a_p = 5.348 \times 10^9$ m $= 0.0357$ AU.
From Kepler's third law:

$$(a_* + a_p)^3 = \frac{G(M_* + M_p)}{4\pi^2} P^2$$

Assuming that the mass of the planet is negligible to the mass of the host star, and that the semimajor axis of the host star is negligible compared to the semimajor axis of the planet gives

$$a_p^3 = \frac{GM_*}{4\pi^2} P^2$$

Solving for a_p, we get

$$a_p = \left(\frac{GM_*}{4\pi^2} P^2 \right)^{1/3} = 5.348 \times 10^9 \text{ m} = 0.0357 \text{ AU}$$

(b) Determine the density of WASP-14b.

Solution: Answer: $\rho = 4843.7 \text{ kg/m}^3$.

To find the mass, we assume the orbit is circular. Since the planet is transiting, its inclination must be approximately 90°. Therefore, the velocity of the host star's movement is the amplitude of the radial velocity curve.

$$v_* = \frac{(-4.02) - (-6.02)}{2} = 0.995 \text{ km/s}$$

The velocity of the planet can be calculated from the semimajor axis from part (a):

$$v_p = \frac{2\pi a_p}{P}$$

Finally, the two velocities v_* and v_p are related by the conservation of momentum: $M_* v_* = M_p v_p$. Solving for M_p gives

$$M_p = \frac{M_* v_*}{v_p} = 1.383 \times 10^{28} \text{ kg}$$

Note that given the eccentricity from part (c), we can find a more accurate mass of the planet; however, assuming circular orbits gives a close approximation.

To find the radius, we compare the magnitude difference during transit. A difference in magnitude Δm relates to flux by

$$\Delta m = -2.5 \log \left(\frac{F_{transit}}{F} \right)$$

The normalized drop in flux during transit (i.e. relative to the normal flux) is therefore

$$\Delta F_{norm} = \frac{F - F_{transit}}{F} = 1 - \frac{F_{transit}}{F} = 1 - 10^{-\Delta m/2.5} = 0.0093$$

The normalized drop in flux relates to the radii of the host star and planet by

$$\Delta F_{norm} = \left(\frac{R_p}{R_*} \right)^2$$

The radius of the planet can then be found: $R_p = R_* \sqrt{\Delta F_{norm}} = 8.786 \times 10^7$ m.

Density is simply

$$\rho = \frac{M_p}{V_p} = \frac{3M_p}{4\pi R_p^3} = 4843.7 \text{ kg/m}^3$$

(c) Determine the eccentricity of the orbit of WASP-14b.

Solution: Answer: $e = 0.0964$.

Because the planet is transiting, we know that the inclination must be 90°. Let ω be the argument of periastron, which is the angle between the periapsis and the plane of the sky. Let θ be the true anomaly, which is the angle between the planet and the periapsis. Therefore, $\theta + \omega$ is the angle between the planet and the plane of the sky.

Let z be the position of the planet along the axis perpendicular to the plane of the sky (i.e. toward and away from the observer). Let r be the distance from the planet to the host star. $z = r\sin(\theta + \omega)$. The radial velocity of the star is simply the time-derivative of z, which is $\dot{z} = \dot{r}\sin(\theta + \omega) + r\dot{\theta}\sin(\theta + \omega)$.

The orbit equation gives the distance from the host star to the planet as a function of true anomaly:

$$r = \frac{a(1-e^2)}{1 + e\cos\theta}$$

Taking the time-derivative of r gives:

$$\dot{r} = \frac{a(1-e^2)}{(1+e\cos\theta)^2} \cdot e\sin\theta = \frac{r\dot{\theta} \cdot e\sin\theta}{1 + e\cos\theta}$$

$\dot{r}$ can then be substituted into the expression for radial velocity:

$$\dot{z} = r\dot{\theta}\left[\frac{e\sin\theta}{1+e\cos\theta} \cdot \sin(\theta+\omega) + \cos(\theta+\omega)\right]$$

From Kepler's second law:

$$\dot{A} = \frac{1}{2}r^2\dot{\theta} = \frac{A}{P} = \frac{\pi a^2\sqrt{1-e^2}}{P}$$

$$r\dot{\theta} = \frac{2\pi a^2\sqrt{1-e^2}}{rP}$$

$$r\dot{\theta} = \frac{2\pi a^2\sqrt{1-e^2}}{P} \cdot \frac{1+e\cos\theta}{a(1-e^2)} = \frac{2\pi a(1+e\cos\theta)}{P\sqrt{1-e^2}}$$

This can then be substituted into the expression for radial velocity:

$$\dot{z} = \frac{2\pi a}{P\sqrt{1-e^2}}\left[e\sin\theta \cdot \sin(\theta+\omega) + (1+e\cos\theta)\cdot\cos(\theta+\omega)\right]$$

$$\dot{z} = \frac{2\pi a}{P\sqrt{1-e^2}}\left[e\cos\omega + \cos(\theta+\omega)\right]$$

The term $\cos(\theta + \omega)$ determines the radial velocity over time, since all other variables are constants. Radial velocity is maximized when $\cos(\theta+\omega) = 1$ and is minimized when $\cos(\theta+\omega) = -1$. Let $k = \frac{2\pi a}{P\sqrt{1-e^2}}$.

$$\begin{cases} k(e\cos\omega + 1) = \dot{z}_{max} \\ k(e\cos\omega - 1) = \dot{z}_{min} \end{cases}$$

Subtracting the two equations gives $2k = \dot{z}_{max} - \dot{z}_{min}$, or $k = \frac{1}{2}(\dot{z}_{max} - \dot{z}_{min})$. Adding the two equations gives $2ke\cos\omega = \dot{z}_{max} + \dot{z}_{min}$. So,

$$e\cos\omega = \frac{\dot{z}_{max} + \dot{z}_{min}}{\dot{z}_{max} - \dot{z}_{min}}$$

$$e = \frac{\dot{z}_{max} + \dot{z}_{min}}{\cos\omega(\dot{z}_{max} - \dot{z}_{min})} = 0.0964$$

2. The star Sualocin (RA: 20^{h} 39.6^{m}, Dec: 15° 54.7', absolute magnitude: -0.4) is about 78 pc away from our solar system, and the star Rotanev (RA: 20^{h} 37.5^{m}, Dec: 14° 35.7', absolute magnitude: 1.6) is about 31 pc away. An alien astronomer is on a planet with Earth's mass and radius orbiting Rotanev. The planet has a uniform albedo of 0.3.
   (a) What is the angular distance between Sualocin and Rotanev?
   (b) What is the distance between these stars in parsecs?
   (c) On the alien's planet, what is the angular separation in the sky between Sualocin and our Sun?
   (d) How much greater is the flux received by the planet from Sualocin than that received from our Sun?

3. Let's suppose that at some point in the recent past, all the hydrogen and helium in the universe had been instantly fused into iron in stars, and that the energy released was thermalized into black body radiation. Take the baryon density to be $\rho_b = 4.2 \times 10^{-31}$ g/cm^3. This is about 75% hydrogen (1 baryon) and 25% helium (4 baryons) by mass. The binding energy per nucleon of $^{56}_{26}$Fe is 8.8 MeV and that of $^{4}_{2}$He is 7.1 MeV.
   (a) What is the current temperature of this black body radiation?

Solution: Answer: $T = 4.40K$.
Without considering intermediary products, we have these two reactions:

$$56\ ^{1}_{1}\mathrm{H} \rightarrow\ ^{56}_{26}\mathrm{Fe}$$

$$14\ ^{4}_{2}\mathrm{He} \rightarrow\ ^{56}_{26}\mathrm{Fe}$$

We can calculate the energy released in each of these reactions by computing the binding energy on each side. There is no binding energy for a hydrogen atom, so the energy released in the first reaction is 56 × 8.8 MeV = 492.8 MeV per Fe. For the second reaction, the binding energy will by (56 × 8.8 MeV) - (14 × 4 × 7.1 MeV) = 95.2 MeV per Fe.
To find how much iron is actually produced, we need to determine the number densities of hydrogen and helium using the mass fraction given in the question:

$$n_H = \frac{0.75\rho_b}{m_H}$$

$$n_{He} = \frac{0.25\rho_b}{m_{He}}$$

where m_H and m_{He} are the masses of hydrogen and helium respectively. Then , the total energy density released is:

$$u = \frac{492.8MeV \times n_H}{56} + \frac{95.2MeV \times n_{He}}{14}$$

To get the temperature, we use:

$$u = aT^4$$

where a is the radiation constant. This gives us a temperature of 4.40 K.

   (b) Determine what wavelength the blackbody spectrum would peak at. What region of the electromagnetic spectrum would this be in?

Solution: Answer: $\lambda = 6.59 \times 10^{-2}$ cm, microwave.
Using Wien's displacement law:

$$\lambda = \frac{b}{T}$$

where $b = 2.898 \times 10^{-3}$ m K, we get a wavelength of 6.59×10^{-2} cm, which is in the microwave region of the electomagnetic spectrum.

(c) How long would it take stars to fuse all the hydrogen and helium in the universe, given that the mean bolometric luminosity per unit volume emitted by stars today is about $3 \times 10^8\ L_\odot/\mathrm{Mpc}^3$? Compare this to the present age of the universe.

Solution: Answer: 7.11×10^{19} seconds (2250 Gyr)
The time it would take is given by the energy density (found in part (a)), divided by the rate of fusion, i.e. the luminosity given in the problem:

$$t = \frac{u}{L}$$

This gives us 7.11×10^{19} seconds, or about 2250 Gyr. This is much longer than the current age of the universe (13.7 Gyr).