# 2022 National Astronomy Competition

## 1 Instructions (Please Read Carefully)

The top 5 eligible scorers on the NAC will be invited to represent USA at the next IOAA. In order to qualify for the national team, you must be a high school student with US citizenship or permanent residency.

This exam consists of 3 parts: Short Questions, Medium Questions and Long Questions.

The maximum number of points is 200.

The test must be completed within 2.5 hours (150 minutes).

Please solve each problem on a blank piece of paper and mark the number of the problem at the top of the page. The contestant's full name in capital letters should appear at the top of each solution page. If the contestant uses scratch papers, those should be labeled with the contestant's name as well and marked as "scratch paper" at the top of the page. Scratch paper will not be graded. Partial credit will be available given that correct and legible work was displayed in the solution.

This is a written exam. Contestants can only use a scientific or graphing calculator for this exam. A table of physical constants will be provided. **Discussing the problems with other people is strictly prohibited in any way until the end of the examination period on March 26th.** Receiving any external help during the exam is strictly prohibited. This means that the only allowed items are: a calculator, the provided table of constants, a pencil (or pen), an eraser, blank sheets of papers, and the exam. No books or notes are allowed during the exam. Exam is proctored and recorded. You are expected to have your video on at all times.

## 2 Short Questions

1. **(10 points)** The energies of an electron in a hydrogen atom are given by

$$E_n = -\frac{13.606 \text{ eV}}{n^2}$$

where $n = 1, 2, 3, \ldots$ represents the principal quantum number of the shell in which the electron is located.

The Ly-$\alpha$ spectral line is produced when an electron transitions from the $n = 2$ to the $n = 1$ energy level. Astronomers observe that the wavelength of the Ly-$\alpha$ line in a distant receding galaxy's emission spectrum is $\Delta\lambda = 7.13$ nm greater than the value measured in a lab.

Calculate the object's approximate distance from us in Mpc (assuming Hubble's constant $H_0 = 70$ km/s/Mpc).

**Solution:** To find the wavelength of the spectral line, we note that

$$\frac{hc}{\lambda} = E_2 - E_1$$

Then,

$$\lambda = \frac{hc}{E_2 - E_1} \approx 121.52 \text{ nm}$$

Answers within $\pm$ 0.5 nm of this result should be accepted due to potential differences in rounding.

In order to calculate the recessional velocity, we use the fact that $\Delta\lambda$ results from the redshift due to motion away from us. Thus,

$$v_r = cz = \frac{\Delta\lambda}{\lambda} c \approx 17600 \text{ km/s}$$

To calculate the distance to this object, we use Hubble's Law:

$$d = \frac{v_r}{H_0} \approx \boxed{251 \text{ Mpc}}$$

2. **(10 points)** The following expression describes the mass function of a binary system:

$$f(M_1, M_2) = \frac{M_2^3 sin^3(i)}{(M_1 + M_2)^2}$$

- $M_1$: Mass of star 1.
- $M_2$: Mass of star 2.
- $i$: Inclination of the orbit.

Consider an **eclipsing** binary system with a period of 70 years and a total semi-major axis of 36 AU. In this system, the semi-major axis of star 1 is two times larger than the semi-major axis of star 2.

Estimate the mass function of the binary system in terms of solar masses.

**Solution:**

The first step is to calculate the total mass of the system:

$$\frac{T^2}{a^3} = \frac{1}{M_{total}}$$

$$M_{total} = \frac{36^3}{70^2}$$

$$M_{total} = 9.5\ M_{\odot}$$

Since the semi-major axis of star 1 is two times larger than the semi-major axis of star 2, the mass of star 2 must be two times larger than the mass of star 1:

$$M_1 + M_2 = M_{total}$$

$$\frac{1}{2}M_2 + M_2 = 9.5\ M_{\odot}$$

$$M_2 = 6.3\ M_{\odot}$$

Since the binary system is eclipsing, the value of the inclination must be extremely close to 90°. Therefore, using this value in the calculations will result in a very accurate value for the mass function:

$$f(M_1, M_2) = \frac{M_2^3 sin^3(i)}{(M_1 + M_2)^2}$$

$$f(M_1, M_2) = \frac{6.3^3 sin^3(90^\circ)}{9.5^2}$$

$$\boxed{f(M_1, M_2) = 2.8\ M_{\odot}}$$

3. **(10 points)** Consider a star $A$ (apparent magnitude $m_A = 10.9$, radius $R_A = 0.42R_{\odot}$). A periodic transiting event is observed to have a decrease the collected flux by 0.07 %. If this event was caused by a transiting exoplanet around star $A$, what would be the radius of that exoplanet in Earth radii?

**Solution:** The depth of a transit $\delta$ equals the ratio of the squared planet to star radii $\frac{R_p^2}{R_*^2}$. So, $R_p = \sqrt{\delta} \times R_* = \sqrt{0.0007} \times 0.42R_{\odot} = 1.212R_{\oplus}$.

4. **(10 points)** Posidonious from the first century BC estimated the circumference of the Earth by observing the rising and setting of the star Canopus. We will retrace his calculations in this problem. He observed Canopus on but never above the horizon at Rhodes. On the other hand, Canopus rose to a maximum of about 7.5° above the horizon at Alexandria. Assume Rhodes and Alexandria have the same longitude and the distance between the two cities is 800 km. Given only this information, estimate the radius of the Earth. How far off is it from the actual value of 6400 km. Justify your answer.

**Solution:** Given the simplifying assumption that the two cities belong to the same longitude, the difference of 7.5 ° in the maximum culmination altitude of Canopus, directly translates to a latitude difference between the two cities.

We then have, if $d$ is the distance between Rhodes and Alexandria,

$$d = \frac{7.5}{360} 2\pi R_E$$

Hence, we get $R_E = 6111\,\text{km}$. This is a difference of about 5 %.

5. **(10 points)** There is an electron with its mass $m_e$ that orbits a proton with mass $m_p$ at a radius $r$. If we only assume the Coloumbic attraction,

   (a) Write an expression of the total energy and the orbital momentum of the electron.

   (b) Rewrite the expression of the total energy $E$ in terms of the orbital momentum $L$, both from the part(a).

   Use $e$ for the electric charge quantity and assume that $m_p$ is incomparably greater than $m_e$ ($m_p \gg m_e$).

**Solution:** (a) The potential energy of the electron is

$$V = -\frac{e^2}{4\pi\epsilon_o r},$$

and the kinetic energy

$$K = \frac{1}{2} m_e v^2.$$

Since the Coloumb force keeps the electron in a centripetal orbit,

$$\frac{1}{2} m_e v^2 = \frac{1}{2} \frac{e^2}{4\pi\epsilon_o r}.$$

Therefore,

$$\begin{aligned} E_{Total} = V + K &= -\frac{e^2}{4\pi\epsilon_o r} + \frac{e^2}{8\pi\epsilon_o r} \\ &= -\frac{e^2}{8\pi\epsilon_o r}. \end{aligned}$$

The orbital momentum is

$$L = m_e v r.$$

(b) From above, we have

$$m_e v^2 = \frac{e^2}{4\pi\epsilon_o r,}$$

Combining with the expression

$$L = m_e v r,$$

we obtain

$$r = \frac{4\pi\epsilon_o L^2}{e^2 m_e}.$$

Therefore,

$$E = -\frac{e^2}{8\pi\epsilon_o}\frac{e^2 m_e}{4\pi\epsilon_o L^2}$$

$$= -\frac{e^4 m_e}{32\pi^2 {\epsilon_o}^2 L^2}.$$

# 3 Medium Questions

1. **(30 points)** *The speed of light*

   The year is 1671 and you are astronomer Ole Rømer, measuring the period of Io's orbit around Jupiter by timing the passages of Io into or out of Jupiter's shadow.

   In December of 1671, Jupiter is at its first quadrature and you observe eclipses at the following times:

   - December 18 at 06:17:48
   - December 20 at 00:46:09
   - December 21 at 19:14:30

   In June of 1672, Jupiter is at its second quadrature and you observe eclipses at the following times:

   - June 19 at 08:42:50
   - June 21 at 03:11:30
   - June 22 at 21:40:10

   (a) **(2 points)** What is the interval between eclipses of Io as measured in December 1671, and what is it in June 1672?

   (b) **(7 points)** These orbital periods are slightly different. Rømer hypothesized that this is evidence that light has a finite speed. Explain why he thought this.

   (c) **(15 points)** Calculate the speed of light from these observations, and what you know about the orbits of Earth and Jupiter. Explain any simplifying assumptions that you make. How close is this speed of light to the correct value? (Hint: at second quadrature Earth is moving directly away from Jupiter, and at first quadrature Earth is moving directly towards Jupiter.)

   (d) **(6 points)** In 1672, Rømer did not have an accurate measurement of the distance from the Earth to the Sun. Write the speed of light *as he would have had to write it*, in terms of the unknown Earth-Sun distance $a$.

   (Be careful: Rømer also did not know the gravitational constant or the mass of the sun!)

**Solution:**

(a) In December 1671, the observed interval between successive eclipses is 42 hours, 28 minutes, and 21 seconds.

In June 1672, the observed interval between successive eclipses is 42 hours, 28 minutes, and 40 seconds.

Notice that the periods differ by 19 seconds.

(b) Since light has a finite speed, we don't observe the eclipses of Io until some time after they actually happen: there is a light delay time. When the distance between Jupiter and Earth is decreasing, the light delay time should be decreasing with each eclipse, so the time between eclipses seems to be shorter. Similarly, when the distance between Jupiter and Earth is increasing, the time between eclipses appears to be longer.

This is very similar to the modern concept of the Doppler effect, though Rømer didn't call it that at the time.

(c) Let the true orbital period be $p$, and the radial velocity of Jupiter relative to Earth be $v$. If the speed of light is $c$, then the delay time between successive orbits will change by $\frac{pv}{c}$. Therefore the observed orbital period is longer than the true orbital period by $\Delta p = \frac{pv}{c}$. The speed of light is thus

$$c = \frac{pv}{\Delta p}$$

Jupiter's orbital velocity is somewhat slower than Earth's, and it is also almost entirely tangential rather than radial. So we make the simplifying assumption that the radial velocity of Jupiter relative to Earth at first and second quadrature is just the Earth's orbital velocity (towards Earth at first quadrature and away from Earth at second quadrature.) So $v$ is just the Earth's orbital velocity, which from Kepler's third law is

$$v = \sqrt{\frac{GM_{\text{sun}}}{a}} = 30 \text{ km/s}$$

The true orbital period of Io is roughly the average of the two measured periods, so $p =$ 42:28:30.5, and $\Delta p = 9.5$ sec. Plugging in these values to the above expression for the speed of light gives

$$c = 4.8 \cdot 10^8 \text{ m/s}$$

Today we know that this is about 60% larger than the true value, which is not bad given all the approximations that were made.

(d) We could try to carry out the same derivation as in the previous part, but leave $a$ as an unknown constant. This gives

$$c = \frac{p\sqrt{GM_{\text{sun}}}}{\Delta p} a^{-1/2}$$

But this expression depends on $G$ and $M_{\text{sun}}$, which Rømer would not have known at the time. Rømer did not know any of $a$, $G$ and $M_{\text{sun}}$ individually, but he did know that the Earth's orbital period

$$T = \frac{2\pi a^{3/2}}{\sqrt{GM_{\text{sun}}}}$$

is one year.

So we can instead write

$$v = \frac{2\pi a}{1 \text{ year}}$$

$$c = \frac{2\pi p}{(1 \text{ year})\Delta p} a = \frac{a}{5\text{m}12\text{s}}$$

This is equivalent to saying that it takes 5 minutes and 12 seconds for light to travel from the Sun to the Earth. It can be argued that Rømer was not really measuring the speed of light, but instead just the light travel time from the Earth to the Sun. (The first accurate determination of the Earth-Sun distance was made a year later by Cassini.)

2. **(30 points)** A meteorite that is radially approaching the Earth collides with a space station that revolves around the Earth in a circular orbit with radius $R$. For all parts of the question, express your results in terms of the mass $M$ of the Earth, the gravitational constant $G$, the mass $m_1$ of the meteorite, and the mass $m_2$ of the space station.

(a) Assume that, after the impact, the meteorite and the space station form a conglomerate that moves in a closed orbit which approaches the center of the Earth at a minimum distance $R/2$. State what the shape of the orbit of the conglomerate is and determine:

(i) the speed of the meteorite just before the collision,
(ii) the minimum and maximum speeds of the conglomerate,
(iii) the maximum distance of the conglomerate from the center of the Earth.

Determine the condition that $m_1$ and $m_2$ must obey so that the aforementioned scenario is possible.

(b) Determine the minimum speed that the meteorite should have just before the collision so that the conglomerate moves in an open orbit after the impact. For this minimum value of the speed of the meteorite, state what the shape of the orbit of the conglomerate would be and determine

(i) the maximum speed of the conglomerate,
(ii) its minimum distance from the center of the Earth,
(iii) the angle traversed by the orbital radius from the moment of the collision until the moment when the conglomerate approaches the center of the Earth to minimum distance.

**Solution:**

(a) The orbit of the conglomerate after the collision will be an **ellipse**.
Conservation of momentum implies that the radial and tangential components ($v_r$ and $v_t$, respectively) of the speed of the conglomerate are:

$$v_r = \frac{m_1}{m_1+m_2} v$$

and

$$v_t = \frac{m_2}{m_1+m_2}\sqrt{\frac{GM}{R}}.$$

Here, $v$ is the speed of the meteorite prior to the collision.
Conservation of angular momentum of the conglomerate during its orbital motion implies

$$\frac{m_2}{m_1+m_2}\sqrt{\frac{GM}{R}}R = v_{max}\frac{R}{2}.$$

Therefore, we deduce that

$$\boxed{v_{max} = \frac{2m_2}{m_1+m_2}\sqrt{\frac{GM}{R}}}.$$

Finally, conservation of mechanical energy per unit mass of the conglomerate yields

$$\frac{1}{2}v_{max}^2 - \frac{GM}{R/2} = \frac{1}{2}(v_r^2+v_t^2) - \frac{GM}{R}.$$

Combining all previous equations, we find

$$\left(\frac{2m_2}{m_1+m_2}\right)^2\frac{GM}{R} - \frac{4GM}{R} = \left(\frac{m_1}{m_1+m_2}\right)^2 v^2 + \left(\frac{m_2}{m_1+m_2}\right)^2\frac{GM}{R} - \frac{2GM}{R},$$

which immediately yields

$$\boxed{v = \frac{\sqrt{3m_2^2 - 2(m_1+m_2)^2}}{m_1}\sqrt{\frac{GM}{R}}},$$

provided that

$$\sqrt{3}m_2 > \sqrt{2}(m_1+m_2) \Leftrightarrow \boxed{m_1 < \left(\sqrt{\frac{3}{2}}-1\right)m_2}.$$

Now, one can find the semi-major axis of the orbit of the conglomerate:

$$\frac{1}{a} = \frac{2}{r_{min}} - \frac{v_{max}^2}{GM} = \frac{4}{R}\Big(1 - \frac{m_2^2}{(m_1+m_2)^2}\Big) \Leftrightarrow a = \frac{R}{4}\frac{(m_1+m_2)^2}{(m_1+m_2)^2 - m_2^2}.$$

Moreover, if we denote the eccentricity of the orbit by $e$, we find

$$a(1-e) = \frac{R}{2} \Leftrightarrow ae = a - \frac{R}{2}.$$

It follows that

$$r_{max} = a(1+e) = 2a - \frac{R}{2} \Leftrightarrow \boxed{r_{max} = \frac{R}{2}\frac{m_2^2}{(m_1+m_2)^2 - m_2^2}}.$$

Conservation of angular momentum immediately yields

$$v_{min} = v_{max}\frac{r_{min}}{r_{max}} \Leftrightarrow \boxed{v_{min} = \frac{2m_1(m_1+2m_2)}{m_2(m_1+m_2)}\sqrt{\frac{GM}{R}}}.$$

(b) The orbit of the conglomerate will have the shape of a **parabola**. This scenario will occur when

$$v_r^2 + v_t^2 = \frac{2GM}{R} \Leftrightarrow \boxed{v = \frac{\sqrt{2(m_1+m_2)^2 - m_2^2}}{m_1}\sqrt{\frac{GM}{R}}}.$$

Conservation of mechanical energy gives

$$v_{max}^2 = \frac{2GM}{r_{min}},$$

while conservation of angular momentum implies

$$v_{max} r_{min} = \frac{m_2}{m_1+m_2}\sqrt{\frac{GM}{R}}R.$$

Therefore,

$$\boxed{v_{max} = \frac{2(m_1+m_2)}{m_2}\sqrt{\frac{GM}{R}}}$$

and

$$r_{min} = \frac{m_2^2}{2(m_1+m_2)^2}R.$$

From the equation of a parabola in polar coordinates,

$$r = \frac{2r_{min}}{1+\cos\theta},$$

one can find

$$\frac{1+\cos\theta}{2} = \cos^2\frac{\theta}{2} = \frac{r_{min}}{r} \Leftrightarrow \theta = 2\arccos\sqrt{\frac{r_{min}}{R}} = 2\arccos\frac{m_2}{\sqrt{2}(m_1+m_2)}.$$

# 4 Long Questions

1. **(45 points)** *The Curious Orbit of James Webb*

   For his upcoming Astrophysics Club presentation, Will researches the recently launched James Webb Space Telescope (JWST), the next-generation telescope designed as the successor of the Hubble Space Telescope. The largest space telescope ever built, the JWST uses its large collecting area to observe in the infrared spectrum. It orbits around the $L_2$ Lagrange point of the Earth-Sun system. Lagrange points are equilibrium points for a small body in the Earth-Sun system; $L_2$ is the point on the Earth-Sun line located beyond Earth's orbit.

   In the problem, let $M$ and $m$ be the mass of the Sun and Earth, respectively, with $M \gg m$. Additionally, consider the Sun's and Earth's radius to be $R_\odot$ and $R_\oplus$ respectively, and the Earth to orbit the Sun in a perfectly circular orbit of radius $R$.

   (a) **(2 points)** The orbit of JWST was designed to circle around $L_2$ in a big enough orbit to avoid Earth's shadow. What is the benefit of i) being at a Lagrangian point and ii) avoiding Earth's shadow?

   (b) **(5 points)** Taking first order approximations, about how far is $L_2$ from Earth? Express your answer both in terms of the variables defined and numerically, in km.

   (c) **(5 points)** In the rotating reference frame in which the Earth and the Sun are stationary, JWST orbits $L_2$ in the plane perpendicular to the Earth-Sun line that passes through $L_2$. If JWST orbits in a circle of radius $r$ around $L_2$ in this frame, what is the minimum $r$ that avoids the Earth's shadow at all times? Express your answer both in terms of the variables defined and numerically, in km.

   (d) **(20 points)** Consider a scenario where JWST is stationary in the aforementioned rotating reference frame and has a small displacement $\boldsymbol{\delta r} = \delta x\hat{\boldsymbol{\imath}} + \delta y\hat{\boldsymbol{\jmath}}$ relative to $L_2$, where $\hat{\boldsymbol{\imath}}$ is the unit vector along the Earth-Sun line away from the Sun and $\hat{\boldsymbol{\jmath}}$ is a unit vector perpendicular to $\hat{\boldsymbol{\imath}}$. Both $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$ are stationary in the rotating frame. To first order (i.e. assuming $|\boldsymbol{\delta r}| \ll x$), what is the acceleration of JWST in the rotating frame?

   (e) **(5 points)** The presence of the Coriolis force in the rotating reference frame destabilizes orbits around $L_1$, $L_2$, and $L_3$ while stabilizing orbits around $L_4$ and $L_5$. **Disregarding the Coriolis force for this part only,** are orbits stable around $L_2$ when there is no Coriolis force? Is this result generalizable? In other words, what can be said about the stability of orbits around an arbitrary, stationary point where there are no masses within the orbit and no fictitious forces involved?

   (f) **(8 points)** Suppose JWST orbits in the circle described in part (c) with a constant speed and an orbital radius of 500,000 km. Suppose further that the jet propulsion of the JWST is programmed to counteract only the Coriolis force; the rest of JWST's motion is due to the natural gravitational dynamics at $L_2$. Using the assumption that the first order expression derived in (d) still applies, if JWST has a mass of 6500 kg, what is the average magnitude of the force over a long period of time? The following averages (calculated from 0 to $2\pi$) might be helpful:

   $$\overline{|\sin x|} = \frac{2}{\pi} \qquad \overline{\sin^2 x} = \frac{1}{2} \qquad \overline{|\sin^3 x|} = \frac{4}{3\pi}$$

   For reference, the magnitude of the Coriolis force is given as,

   $$|\boldsymbol{F}| = 2m|\boldsymbol{\omega} \times \boldsymbol{v}|$$

**Solution:**

(a) i) A Lagrangian point is stationary with respect to the Earth and the Sun, which is beneficial for a space telescope because it requires little fuel to maintain its position.

ii) Avoiding Earth's shadow is beneficial for JWST because it always has access to the Sun's energy, never to be interrupted by eclipses.

(b) This is a classic problem. $L_2$ is the Lagrangian point just beyond Earth's orbit. Let us define the $L_2$-Earth distance to be $x$, where $x \ll R$. Taking into account the forces of the Earth and the Sun yields:

$$\frac{GM}{(R+x)^2} + \frac{Gm}{x^2} = \omega^2(R+x)$$

where $\omega = \sqrt{GM/R^3}$ is the rotational velocity of Earth's orbit. Taking advantage of the binomial approximation $(1+y)^\alpha \approx 1 + \alpha y$ when $y \ll 1$, we can simplify our expression to the following:

$$\frac{GM}{R^2}\left(1 - \frac{2x}{R}\right) + \frac{Gm}{x^2} \approx \frac{GM}{R^2}\left(1 + \frac{x}{R}\right)$$

Solving the equation yields

$$\boxed{x = R\sqrt[3]{\frac{m}{3M}} = 1.50 \times 10^6 \text{ km}}$$

(c) From geometry, JWST must lie outside of the cone that is defined by the common internal tangents of the Earth and the Sun. The tip of the cone has an angle

$$2\theta = 2\sin^{-1}\left(\frac{R_\odot + R_\oplus}{R}\right)$$

If you extend one of those internal tangents as follows (insert diagram), one can see that $r_{\min}$ satisfies the following equation:

$$\tan\theta = \frac{R_\odot/\cos\theta + r}{R+x}$$

Therefore:

$$\boxed{r_{\min} = \frac{(R+x)R_\oplus + xR_\odot}{\sqrt{R^2 - (R_\odot + R_\oplus)^2}} = 1.34 \times 10^4 \text{ km}}$$

Note that realistically, $R_\oplus, R_\odot \ll R$ and $x \ll R$, so $r_{\min} \approx R_\oplus + xR_\odot/R$, which is off by less than 1%.

(d) Because $L_2$ is a Lagrangian point, we know that in the rotating frame, the forces at $L_2$ balance. Therefore, we only need to find first order changes in force relative to $L_2$. Suppose a mass $M$ is located at a distance $R$ to the left of $L_2$. Then the change in gravitational field from $L_2$ to $L_2 + \boldsymbol{\delta r}$ is

$$\boldsymbol{\delta g} = -\frac{GM}{((R+\delta x)^2 + \delta y^2)^{3/2}}((R+x)\hat{\boldsymbol{\imath}} + \delta y\hat{\boldsymbol{\jmath}}) + \frac{GM}{R^2}\hat{\boldsymbol{\imath}}$$

The $\delta y^2$ in the denominator of the first term is of second order and can be discarded:

$$\boldsymbol{\delta g} \approx -\left(\frac{GM}{(R+\delta x)^3}(R+x) - \frac{GM}{R^2}\right)\hat{\boldsymbol{\imath}} - \frac{GM}{R^3}\delta y\hat{\boldsymbol{\jmath}}$$

The binomial approximation can be used once again:

$$\boldsymbol{\delta g} \approx \frac{GM}{R^2}\left(1 - \frac{2x}{R} - 1\right)\hat{\boldsymbol{\imath}} - \frac{GM}{R^3}\delta y\hat{\boldsymbol{\jmath}} = \frac{GM}{R^3}(2\delta x\hat{\boldsymbol{\imath}} - \delta y\hat{\boldsymbol{\jmath}})$$

You might notice that this expression looks similar to that of tidal forces, because what we've done here is exactly the same! Now, to find the acceleration of JWST, we simply need to add the $\boldsymbol{\delta g}$ of the Earth and the Sun, then add $\boldsymbol{\delta a}$ of the centrifugal force on top (the coriolis force is 0 since JWST is stationary). $\boldsymbol{\delta a}$ due to the centrifugal force is simply:

$$\boldsymbol{\delta a} = \delta(\omega^2\boldsymbol{r}) = \omega^2\boldsymbol{\delta r} = \frac{GM}{R^3}(\delta x\hat{\boldsymbol{\imath}} + \delta y\hat{\boldsymbol{\jmath}})$$

Therefore, the acceleration of JWST is

$$\boldsymbol{a} = \left(\frac{GM}{(R+x)^3} + \frac{Gm}{x^3}\right)(2\delta x\hat{\boldsymbol{\imath}} - \delta y\hat{\boldsymbol{\jmath}}) + \frac{GM}{R^3}(\delta x\hat{\boldsymbol{\imath}} + \delta y\hat{\boldsymbol{\jmath}})$$

Plugging in $x = R\sqrt[3]{m/3M}$ and sticking to first order in $\boldsymbol{\delta r}$ yields

$$\begin{aligned}\boldsymbol{a} &\approx \left(2\left(\frac{GM}{R^3} + \frac{3GM}{R^3}\right) + \frac{GM}{R^3}\right)\delta x\hat{\boldsymbol{\imath}} + \left(-\left(\frac{GM}{R^3} + \frac{3GM}{R^3}\right) + \frac{GM}{R^3}\right)\delta y\hat{\boldsymbol{\jmath}} \\ &= \boxed{\frac{GM}{R^3}(9\delta x\hat{\boldsymbol{\imath}} - 3\delta y\hat{\boldsymbol{\jmath}})}\end{aligned}$$

(e) From our expression in part (d), we see that a small displacement in the $\hat{\boldsymbol{\imath}}$ direction results in a force that points in the same direction, meaning any orbits are $\boxed{\text{unstable}}$ in that direction.

In fact, this result is general for all orbits around arbitrary points with no mass and no fictitious forces. Using Gauss's law for gravitation, the flux through a surface surrounding such a point must be 0, indicating it is impossible for the gravitational force to be pointing inward for displacements in every possible direction.

(f) The acceleration in the direction perpendicular to $\hat{\boldsymbol{\imath}}$ was found in part (d) to be $-3GM\delta y/R^3$. Setting this equal to the centripetal acceleration, we have

$$\begin{aligned}\frac{3GMr}{R^3} &= \frac{v^2}{r} \\ v &= r\sqrt{\frac{3GM}{R^3}}\end{aligned}$$

The magnitude of the Coriolis force is

$$|\boldsymbol{F}| = 2m|\boldsymbol{\omega} \times \boldsymbol{v}| = 2m\sqrt{\frac{GM}{R^3}}r\sqrt{\frac{3GM}{R^3}}|\sin\theta|$$

where $\theta$ is the angle between $\boldsymbol{v}$ and $\boldsymbol{\omega}$. Since it varies from 0 to $2\pi$, we use the given value of $\overline{|\sin\theta|}$ to find the average magnitude to be

$$\boxed{|\boldsymbol{F}| = \frac{4\sqrt{3}GMmr}{\pi R^3} = 0.28 \text{ N}}$$

Note just how small the force is!

2. **(45 points)** *The Sundial I*

While on a walk in Princeton University, Leo stumbled upon the following sundial, mounted on the southern wall of a building:


Figure 1: Picture taken by Leo Yao, December 2020.

He was familiar with the lines pointing outwards from the center, marking off time of day. However, he also noticed the three curves crossing the other lines. After a bit of thought, he realized these curves marked off the path of the shadow on the equinoxes and solstices.

(a) **(3 points)** For each of the two equinoxes and solstices, match the day to the curve (top, middle, bottom) denoting the path of the shadow on that day. (No explanation needed)

(b) **(1 point)** For the days corresponding to the middle curve, what is the declination of the Sun on those days? Assume the length of the day is small compared to the length of the year. (No explanation needed)

He then noticed that the middle curve seemed to be a straight line, and started thinking about if this is the case. He first considered a simpler system: a stick mounted vertically on a flat surface, casting a shadow on flat ground.

(c) **(9 points)** Consider the shadow of the tip of the stick, which might possibly trace a straight line over the course of the day. Explain why, if this happens, it can only happen on a day when the Sun's declination is that determined above.

This part can be solved independently, or as part of your solution for the next part. If your solution for the next part also proves this, note that down on your solution sheets, and proceed directly to the next part.

(d) **(24 points)** Prove that, for the Sun's declination determined above, the shadow of the tip of the stick traces a straight line over the course of the day.

Any method is acceptable, as long as it is presented clearly and rigorously. For example, one possible method might involve the following steps:

i) Determining the orientation of the line and explaining why it must be in this orientation;
ii) Determining the length of the shadow for a given position of the Sun in alt-az coordinates;
iii) Deriving a relation between altitude and azimuth given that the tip of the shadow is on the line;
iv) Determining a constant quantity and showing that it is constant over all positions of the Sun that day.

If you skipped the previous part, make sure your proof also shows the inverse: that for a different declination of the Sun, the tip's shadow does not trace a straight line.

You do not necessarily need to follow these steps. Simpler and/or faster methods may be possible, including those that do not need any equations. **Any fully-formed, valid explanation gives full credit.**

After figuring out the simpler case, Leo realized that he could easily generalize it to the sundial mounted on the wall. He then started thinking about other ways the model and the sundial on the wall differed, and thought about the orientation of the center line, noticing that it was not perfectly horizontal, but instead slanted.

(e) **(8 points)** Based on the picture and the orientation of the center line, does the wall run perfectly East-West, or does it run Northeast-Southwest or Southeast-Northwest? Explanation needed for credit.

Assume the wall is perfectly vertical. **This part can be solved independently of the previous two parts.**

**Solution:**

(a) The Sun is highest on the summer solstice, and lowest on the winter solstice. Therefore shadows for sundials on the ground are shorter in summer than in winter. However, this sundial is on a vertical wall, so this trend is actually reversed: the top curve (shortest shadows) corresponds to the winter solstice and the bottom curve (longest shadows) corresponds to the summer solstice.

Also notice the visible shadow close to the top curve; as the picture was taken in December 2020, this shadow corresponds to the winter solstice and again the top curve.

The equinoxes therefore correspond to the curve in the middle.

(b) On the equinoxes, the declination of the Sun is $\delta = 0$.

(c) There are many possible solutions to this part; here we present one.

First note that, as the Sun's path is symmetric across the meridian, by symmetry, the line must run perfectly East to West.

Consider the Sun's position at solar noon on any day; if the Sun does not rise, no line is possible. But if the Sun does rise, the altitude of the Sun as it crosses the meridian is nonzero. For a finite stick height, the shadow length is finite. Therefore, the line must be a finite distance from the stick.

Now consider the azimuth of the Sun at sunset. If $\alpha \neq 270$, then there is some component of the shadow that runs North-South. But as the Sun's altitude goes to 0, the shadow length goes to $\infty$. As the line runs East-West, the line must therefore be an infinite distance from the stick, which is a contradiction.

Therefore, this is only possible for $\alpha = 270$ at sunset, implying $\delta = 0$.

(d) We present a variety of solutions. First we present the method described in the problem statement:

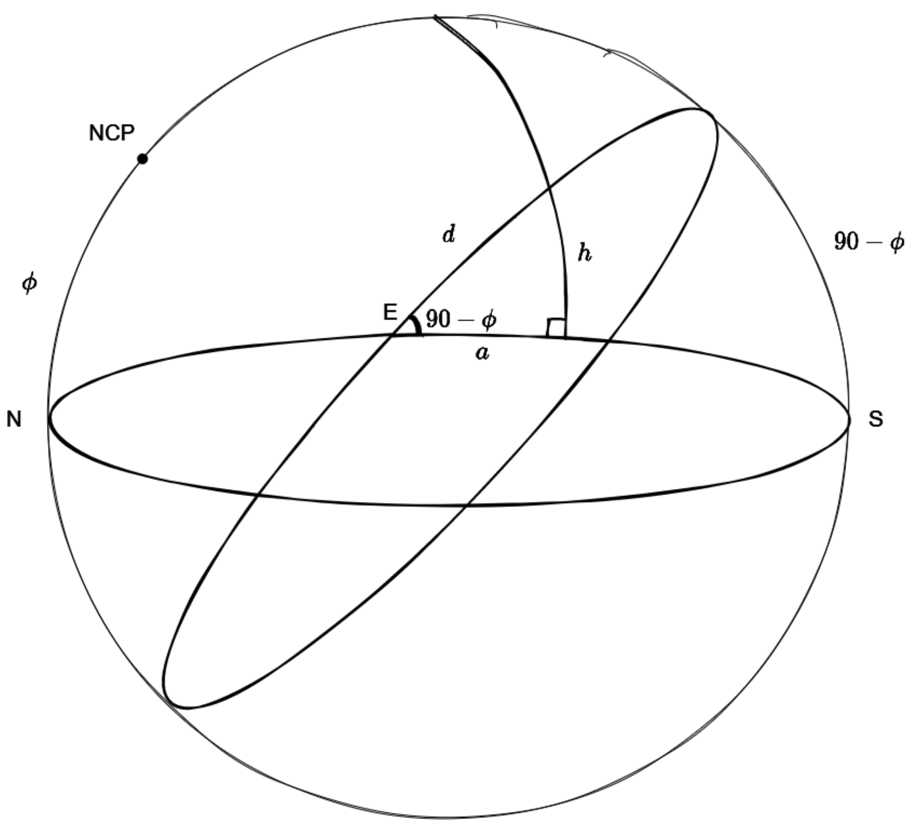
i) From the previous part, we know the line must run perfectly East to West, at a finite distance from the stick.

ii) If the Sun has altitude $h$, then if the stick has length $h_0$, then the length of the shadow is given by $l = \frac{h_0}{\tan h}$.

iii) Let $a = \text{azimuth} - 90$, such that the Sun rises at $a = 0$ and sets at $a = 180$. Let the line be a distance $x_0$ from the stick; then we need $l \sin a = x_0$.

iv) Therefore, $\frac{h_0 \sin a}{\tan h} = x_0$, which we need to hold for all positions $(a, h)$ of the Sun over the entire day. We have therefore derived the constant quantity $\frac{\sin a}{\tan h}$.
To finish off the proof, we need to show this is a constant quantity:

$$\frac{\sin a}{\tan h} = \frac{\sin a \cos h}{\sin h}$$




Using the spherical law of cosines:

$$= \frac{\sin a}{\sin h} \left( \cos a \cos d + \sin a \sin d \cos(90 - \phi) \right)$$

Using the spherical right triangle equality $\cos d = \cos h \cos a$:

$$= \frac{\sin a}{\sin h}\left(\cos^2 a \cos h + \sin a \sin d \sin\phi\right)$$

$$= \cos^2 a \frac{\sin a \cos h}{\sin h} + \sin^2 a \frac{\sin d}{\sin h}\sin\phi$$

Noticing the first term has the same form as the original:

$$(1-\cos^2 a)\frac{\sin a}{\tan h} = \sin^2 a \frac{\sin d}{\sin h}\sin\phi$$

Using $1-\cos^2 a = \sin^2 a$ and cancelling terms:

$$\frac{\sin a}{\tan h} = \frac{\sin d}{\sin h}\sin\phi$$

Using the spherical law of sines $\frac{\sin h}{\sin(90-\phi)} = \sin d$:

$$= \frac{\sin\phi}{\sin(90-\phi)} = \tan\phi$$

Therefore, this is a constant quantity, which means the tip of the shadow is always on this straight line.

If the previous part was skipped, it is also necessary to prove the line is perfectly East to West, and that it is a finite distance from the stick, as those assumptions were used in the derivation above.

The inverse would also need to be correctly justified. For example, one way to do this would be to simplify $\frac{\sin a}{\tan h}$ without the $\delta = 0$ assumption.

Using the spherical law of cosines to derive the coordinate transformation:

$$\cos(90-\delta) = \cos(90-\phi)\cos(90-h) + \sin(90-\phi)\sin(90-h)\cos(a+90)$$

$$\sin\delta = \sin\phi\sin h - \cos\phi\cos h\sin a$$

$$\sin a = \frac{\sin\phi\sin h - \sin\delta}{\cos\phi\cos h}$$

Applying this transformation:

$$\frac{\sin a}{\tan h} = \frac{1}{\tan h}\frac{\sin\phi\sin h - \sin\delta}{\cos\phi\cos h}$$

$$= \frac{\sin\phi\sin h}{\cos\phi\sin h} - \frac{\sin\delta}{\sin h\cos\phi}$$

$$= \tan\phi - \frac{\sin\delta}{\cos\phi}\frac{1}{\sin h}$$

The latitude $\phi$ is a constant, and the declination $\delta$ is approximately unchanged over the course of a day. However, $\frac{1}{\sin h}$ varies over the course of a day as the Sun rises and sets. Therefore, for this to be a conserved quantity, we need $\sin\delta = 0$, implying $\delta = 0$.

However, this problem can be solved much faster geometrically. A simple solution using nothing but properties of great circles and planes goes as follows:

Place the tip of the stick at the center of the celestial sphere. On the equinoxes, the declination of the Sun is 0, so the path of the Sun is a great circle; therefore the path of the Sun through the day and the tip of the stick lie in the same plane. Therefore, the shadow of the tip of the stick must be in the same plane. Intersecting this plane with the plane of the ground gives a straight line for the shadow's path. □

Notice that this solution quickly generalizes to any other intersecting plane, be it a vertical wall or any other orientation the sundial is mounted in.

(e) Consider two positions of the Sun on the equinox, equidistant from noon. By symmetry, the altitudes are the same, and the aziumths are equally offset from due South.

Consider the shadow cast by the Sun at each of these times, which is the intersection of the line connecting the Sun's position and the tip of the sundial with the wall. As the Sun is up in the South, these lines slope down as they head North.

Looking at the line in the picture, the line traveling West drops less compared to the line traveling East. This implies the wall juts out towards the observers perspective on the left, meaning the wall is angled Southwest to Northeast.

Looking at the actual building, Fisher Hall, on Google Maps, it indeed is oriented that way!

Problem adapted and extended from Roberto Bozcko and Lucas Carrit Delgado Pinheiro.